The Science of Ballistics: Mathematics Serving the Dark Side

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ABSTRACT

Ballistics, rooted as it is in the study of motion and indispensable as it has become in the affairs of nations, has held the interest of scientists, engineers, generals, and rulers. Fundamental contributions to this science, including many of the most significant contributions, have been made by mathematicians. This paper highlights some of these contributions, especially those central to the early development of ballistics, as well as their social, political, and technological contexts.

La balistique, enracinée telle quelle dans l'étude de la mouvement, et aussi indispensable comme elle a devient dans les affaires entre les nations, a retenu l'intérêt des scientistes, des ingénieurs, des générales, et des chefs. Contributions fondamentales à cette science, y compris beaucoup des contributions les plus significatifs, ont été faire par les mathematiciens. Cette article illumine quelques unes de ces contributions, spécialement celles principales en voie de développement de la balistique, ainsi que leur contextes sociaux, politiques, et technologiques.

1. Beginnings of a New Science

Ballistics, a word apparently originating in the text *Ballistica et Acontismologia* (1644) by Marin Mersenne [12, p. 106], comprises both interior ballistics, the study of the chemistry and thermodynamics that occurs in the barrel of a gun, and exterior ballistics, the study of the motion of an object projected from a gun [22]. This paper will consider primarily exterior ballistics, the branch most indebted to mathematics.

Exterior ballistics is founded on the physics of a projectile as it moves through air. In Aristotle's physics, motion is of two types: natural motion, such as that of a falling stone or rising smoke, is resisted by the medium such as air or water in which it occurs; violent motion results from applying a force to (e.g. pushing or throwing) an object and is assisted by the medium—indeed, violent motion cannot exist without a medium [12, p. 80]. In the fourteenth century, philosophers Jean Buridan and Nicole Oresme challenged these ideas, asserting that both types of motion are resisted by the medium and that violent motion persists owing to the *impetus* imparted by the force initially applied; the impetus of an object depends on both the weight of the object and the force initially applied to it (similar to the modern concept of momentum, the product of mass and velocity); they also parted with Aristotle in stating that a heavy object acquires more impetus and thus accelerates as it falls [12, pp. 80-82]. Albert of Saxony, another philosopher from this period, regarded the trajectory of a projectile as having three parts (see Figure 1): the first part is an oblique line on which impetus dominates the motion; the second is a curve on which impetus is reduced due to resistance; the final part is a vertical line on which natural motion prevails [12, pp. 81-83].

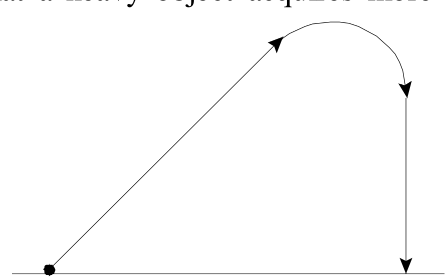

Figure 1

Impetus ideas were part of the intellectual climate of sixteenth century Europe and certainly influenced Niccolo Fontana, better known as Tartaglia, and Galileo [12, pp. 81-83]. Tartaglia, a mathematician famous for his priority dispute with Girolamo Cardano over the formula for the roots of the general cubic equation, published in 1537 the first text on ballistics, *Nova Scientia*, in three short books, which enjoyed great success well into the second half of the seventeenth century [12, pp. 36-37]. The popularity of this work stemmed largely from its conservatism: it did not seriously deviate from either the predominant philosophical opinions of the time (Tartaglia's trajectory is essentially that of Figure 1, two straight lines joined smoothly by the arc of a circle, although it was an idealization since he conceded that its initial part is negligibly curved) or the views of experienced artillerists, and he couched his arguments in the familiar mathematical language of proposition and proof (e.g. in Book 2, Proposition 7 asserts the proportionality of all trajectories with the same elevation angle, and Proposition 8 states that the maximum range obtains at a launch angle of 45°) [12, pp. 36-42]. Tartaglia often shows a keen awareness of the socio-political context of his work: in the dedication of *Nova Scientia*, he writes that "I fell to thinking it a blameworthy thing … to study and improve such a damnable exercise, destroyer of the human species … But now, seeing that the wolf [Ottoman sultan Suleiman I] is anxious to ravage our flock, … it no longer appears permissible to me at present to keep these things hidden" [6, pp. 68-69], and he claims in his *Quesiti et Inventioni Diverse* (1546) that he can, for any cannon, "set up a table of all the shots which that gun will shoot, at any elevation" and that this knowledge can convey a "secret … kept with the one who has got the table" [4].

To our modern eyes, Tartaglia's ballistics looks medieval and arbitrary compared to Galileo's simple and elegant theory. It must be noted, however, that parts of Tartaglia's theory (e.g. the two propositions mentioned above) are consistent with Galileo's and that Tartaglia attempted to incorporate all the complexity of air resistance, the neglect of which is the foundation of Galileo's theory. The law of falling bodies (whereby the vertical distance fallen from rest is proportional to the square of the time falling) [8, p. 166], the undiminished horizontal speed of a body [8, p. 197], the independence of vertical and horizontal motion for a projectile, and its consequent parabolic path [8, p. 217], all follow (when gravity is constant and vertically downward) from the absence of resistance. The path of a real cannonball through air looks as much like Tartaglia's trajectory as Galileo's parabola, but Galileo's neglect of air resistance was the abstraction needed for the science of motion to advance. Nevertheless, although he acknowledges the challenge of air resistance for his theory [8, pp. 224-229], Galileo minimizes the significance of it for ballistics: he writes, for example, that the "excessive impetus of violent shots can cause some deformation in the path of a projectile … But this will prejudice our Author [Galileo] little or nothing in practicable operations" and that shots "made with mortars charged with but little powder … trace out their paths quite precisely" [8, p. 229]. In this respect, Galileo was certainly mistaken, as shown in §3. Galileo's table [8, p. 251] giving the semi-range and altitude of shots fired at the same initial speed and various elevation angles (the arbitrary scale used for these distances is fixed, owing to the proportionality of all Galilean trajectories with the same elevation angle, by one measurement at any angle for a specific cannon, ball, and charge) further suggests that he conflated his theory and real-world ballistics and that he was alert to socio-political forces (such as his patrons, the Medici family).

The tension between theory and practice is illuminated sharply by some correspondence on ballistics involving Evangelista Torricelli, arguably Galileo's most accomplished student. In 1644, Torricelli published *Opera Geometrica*, the second book of which clarifies and extends Galileo's work on ballistics; it includes new constructions (e.g. of a parabolic trajectory), new methods (e.g. for the calculation of the range when a shot is fired uphill), and new expressions (e.g. for the range under various circumstances) [12, pp. 91-95]. In the case of a cannon fired at point-blank (zero elevation angle) and mounted at height h, the range is expressed (in modern notation) as $\sqrt{2hR}$, where R is the maximum range (i.e. the range for an elevation angle of 45°) when the cannon is not mounted [12, p. 94]. In a letter to Torricelli from 1647, a certain Giovanni Renieri wrote, "Your work on the motion of projectiles … has reached Genoa and given our gentlemen the opportunity to make several experiments with various kinds of guns" [21]. One of these experiments involved mounting a gun at a height of 2 ells (about 2 paces) and determining that its maximum range was about 2300 paces while its point-blank range was about 400 paces, far larger than the value (about 96 paces) given by Torricelli's expression; Renieri "was astonished that such a well-grounded theory turned out to work so badly in practice" [21]. In his reply, Torricelli first stated that his ballistics is "ex hypothesi", based on mathematical assumptions about the horizontal and vertical motion of a projectile, and may not reflect physical reality; he went on to enumerate possible reasons for the unexpected result, the main one being air resistance (which could well account for much of the result, since it would shorten the maximum range far more than the point-blank); in reply to a later letter from Renieri lamenting another unexpected outcome, Torricelli emphasized that *Opera Geometrica* was, as stated in the text itself, intended for philosophers, not gunners [21]. Why then, Renieri might well have asked, does the text talk so freely of guns, shots, and walls to be hit; why does it present the design (complete with diagram) of a new gunner's quadrant giving altitude and range [12, p. 95-99]?

In a sense, mathematics is the source of Torricelli's ambivalence, just as it is also both a strength and a weakness of Galileo's theory. Mathematical reasoning makes the theory appealing and convincing, a logical necessity, and confers on it mathematical precision. But it is this very precision that renders Galilean ballistics so easily falsifiable: careful measurements soon reveal its dreadful inaccuracy. Repeatable test firings were difficult to achieve in the seventeenth century, with its poor manufacturing quality and lack of standardization (e.g. for cannon, shot, and powder), unstable gun platforms, and smooth-bore guns [12, p. 55, 70]. However, even without these challenges, the problem of air resistance still remained. Galileo's theory can be applied tolerably well to a baseball tossed casually in a backyard, but at the speed of a ball emerging from the muzzle of a cannon, it is totally inadequate.

2. Accounting for Air Resistance

Galileo wrote that a “disturbance arises from the impediment of the medium … [which] is incapable of being subjected to firm rules” [8, p. 224], but he also claimed that “the amount of speed of the moveable is itself both the cause and the measure of the amount of resistance” [8, p. 227], suggesting a rule that the magnitude of air resistance for a moving body is proportional to its speed. In any case, the experimental results Galileo cited as evidence for this rule—two identical simple pendulums, one with a large initial amplitude and the other a small one, locked in identical rhythm for hundreds of periods [8, pp. 226-227]—could not have actually occurred, since the period of a simple pendulum (in air or in a vacuum) increases, slightly but noticeably, with its angular amplitude. Nevertheless, this rule was a step in the right direction, and Galileo used a similar experiment involving two simple pendulums of the same length, one with a bob of cork and the other of lead, to demonstrate that the effects of air resistance increase as the density of a moving body decreases [8, pp. 87-88]. Galileo also made the keen observation that a body falling through air does not accelerate forever but instead reaches a terminal speed at which gravity and air resistance are balanced [8, p. 228].

Imaginative work on the resistance of fluids was done during 1668-9 by Christaan Huygens, who had earlier (1664) advanced ballistics by determining a precise value of g, the acceleration of gravity, using a simple pendulum [12, pp. 111-119]. Huygens’ experimental research on resistance was particularly impressive. One of his experiments involved a block of wood pulled through a long trough of water by a weight attached to the block by a cord passing over a pulley—it was found that the maximum speed reached by the block increased approximately as the square root of the weight, indicating that the resisting force of the water was roughly proportional to the square of the block’s speed; another experiment involving wheeled carts carrying screens to induce air drag, though less conclusive, suggested that the resistance of air also satisfied a square law of this kind [12, p. 115]. Results such as these convinced Huygens that he had discovered a single law of fluid resistance, a law that could be applied even to cannon balls moving at very high speeds [12, pp. 115-116]; this generalization fails, as will be seen below. Huygens’ theoretical work on resisted motion was also remarkable: he deduced formulas describing vertical motion subject to air resistance varying as the speed and as the square of the speed, and devised a construction for the trajectory of a body projected at 45° and resisted in proportion to its speed (he was unable to do the same for resistance proportional to speed squared) [12, pp. 113-117]. Huygens’ work on fluid resistance was not published until 1690 as an addition to his *Discours de la Pesanteur* [12, p. 147].

It seems poorly known that Book 2 of Isaac Newton’s *Principia* [19], comprising three books, is devoted largely to the study of motion in a resisting fluid, and that Book 2 includes the descriptions of extensive and detailed experiments to determine the nature of fluid resistance. The first section of Book 2 deals with the motion of a body resisted in proportion to its speed, culminating in a construction and analysis (Proposition 4 and corollaries) of the trajectory of a projectile resisted in this way [19, pp. 636-640], a special case of which was treated by Huygens, as noted above; a modern expression of this trajectory is given in §3. The section concludes with a short scholium in which

Newton states that "the hypothesis that the resistance … is in the ratio of the velocity belongs more to mathematics than to nature"; he goes on to say that resistance is proportional to the square of the speed in fluids "wholly lacking in rigidity", justifying this by saying essentially that a body moving at a greater speed through such a fluid produces proportionally greater velocities in proportionally more particles of the fluid per unit time, and so the change per unit time in the total momentum of the fluid, and hence its resisting force (by Newton's second and third laws of motion), is greater in proportion to the square of the body's speed [19, p. 641].

Section 2 of Book 2 deals with motion resisted as the speed squared, concluding with Newton setting himself the task (Proposition 10) of finding, given any curve, how the density of a fluid medium must vary along that curve so that a projectile follows the curve as it moves under the action of gravity and the medium's resistance, assumed to vary jointly as the medium's density and the square of the projectile's speed; this task is a kind of *inverse problem* of interest to Newton probably because he was unable to solve directly for the projectile's trajectory under the square law of resistance in a uniformly dense medium. After a general analysis of the problem, four example trajectories are considered: a semi-circle, a parabola, a hyperbola (with a slant asymptote and a vertical asymptote for its ascending and descending parts, respectively), and a generalized hyperbola (for which the vertical distance from its slant asymptote varies as an arbitrary inverse power of the distance from its vertical asymptote—this inverse power is unity for a standard, conic hyperbola) [19, pp. 655-664]. From these examples, Newton infers that the trajectory of a projectile in a uniformly dense medium "approaches closer to these hyperbolas than to a parabola" [19, p. 664], and he then states eight rules that describe and determine hyperbolas (both conic and generalized) approximating this trajectory under various (e.g. initial) conditions. The similarity between Newton's hyperbolic trajectories, with their asymptotes, and the trajectory depicted in figure 1 is quite remarkable.

In the third section of Book 2, Newton examines some cases of rectilinear motion for which resistance is a linear combination of the speed and the square of the speed. In a scholium at the end of the section [19, pp. 678-679], Newton asserts that the "resistance encountered by spherical bodies in fluids arises partly from the tenacity, partly from the friction, and partly from the density of the medium", and he indicates that these three components of resistance correspond to terms that are constant, proportional to the speed, and proportional to the square of the speed, respectively. George E. Smith, in an essay on Book 2 in [19, pp. 188-194], compares this representation of the force of fluid resistance,

$$F_{resist} = a + bV + cV^2 , \tag{1}$$

where V is the speed and a, b, and c constants, to the modern formulation

$$F_{resist} = \tfrac{1}{2} C_D \rho A_f V^2 , \tag{2}$$

with C_D the non-dimensional drag coefficient, ρ the fluid's density, and A_f the frontal area of the body resisted by the fluid. Newton understood the coefficient c in (1) to vary as ρA_f [19, p. 732, 744], which is largely consistent with (2), but Smith points out that Newton was mistaken to include the constant a in (1) [19, p. 189] and that "any approach that treats resistance as the superposition of two terms, one varying as V and the other as

V^2, misrepresents the physics" since the drag coefficient for a given body and medium can vary with the body's speed, which affects the nature of the flow around the body and whether the fluid's internal friction (viscosity) or its inertia (which arises from its density) dominate the flow [19, p. 193].

Sections 6 and 7 of Book 2 contain material unique in Newton's *Principia*: extensive descriptions of experiments and their results. Section 6 begins with a number of propositions about the oscillation of simple (and generally cycloidal) pendulums in non-resisting and resisting mediums [19, pp. 700-712], and experiments described at the end of the section [19, pp. 713-722] involving pendulums (of wood, lead, or iron) in several fluids (air, water, mercury) use these propositions to investigate fluid resistance by observing the decay of oscillations. In Section 7, Newton uses hypothetical models of resistance to derive, essentially, expressions for the coefficient c in (1) for both a spherical projectile and a cylindrical one moving parallel to its axis, and he also attempts to determine the axisymmetric shape of least resistance [19, pp. 724-749]. Although these theoretical results did not withstand the scrutiny of later investigators, Smith asserts [19, p. 192] that the experiments described later in the section [19, pp. 750-759], the results of which Newton compared with those of his theory, yielded "the first accurate measures of resistance forces" in that the values for the drag coefficient C_D of a sphere that can be extracted from these experiments are in good agreement with current values. These experiments involved spherical balls dropped in water and in air, the latter from the top of St. Paul's Cathedral in London and employing, in the final experiment led by Newton's colleague J. T. Desaguliers in 1719, hogs' bladders inflated with air. Newton reports [19, p. 760] that these experiments produced more accurate and lower values for resistance than his experiments with pendulums, owing to the fluid motion induced by a pendulum that opposes it on its return swing and to the drag on its suspending cord.

Despite Newton's remark that his work on the axisymmetric shape of least resistance may "be of some use for the construction of ships" [19, p. 730], the tone of Book 2 of the *Principia* is one of pristine disregard for the application of its subject to ballistics. British mathematician and engineer Benjamin Robins, however, was keen to use the principles of Newtonian mechanics in this area [22]. Robins' invention of the ballistics pendulum (ca. 1742), a device whose amplitude measures the speed of a projectile that strikes its bob, advanced the science of ballistics in two ways: it allowed the muzzle speed of guns to be accurately determined and, by measuring the speeds of shells at various distances from the gun firing them, enabled the air resistance of shells to be calculated [22]. Using both experimentation and theory, Robins investigated a number of phenomena associated with projectiles: he measured the lateral deflection of musket balls by firing at tissue paper curtains and locating bullet holes, correctly concluding that the spin on a ball induced by contact with the musket barrel pushed the ball in the direction of increased air flow (generally called the *Magnus effect*, after physicist H. G. Magnus who investigated it a century later, but sometimes called the *Robins effect*); he confirmed (with the assistance of another invention, the whirling arm) that air resistance varies roughly as the square of the speed for slow-moving projectiles but observed a sharp increase in resistance (by about a factor of 3) at supersonic speeds, the first anticipation of the *sound barrier* and a result that contradicted parts of Huygens' and Newton's theories of fluid resistance (the coefficient C_D in equation (2) above is actually a function of the *Mach number* [5, pp. 58-68] or *Reynolds number* [17]); he developed a theory of interior ballistics that predicted the work done on a musket ball by combustion gases in the barrel and the resultant muzzle speed, which compared favourably with the actual speed determined with a ballistics pendulum [22]. One indication of Robins' influence in the eighteenth century and the esteem others had for his work is the German translation of Robins' *New Principles of Gunnery* (1742) produced in 1745 by Leonard Euler, who had been asked by Frederick the Great to translate the best available text on ballistics theory for the education of his artillery officers; another indication is the twelve-page summary of *New Principles* produced in 1788 by young artillery lieutenant Napoléon Bonaparte [22].

Robins work was foundational for further research in experimental ballistics, but his ballistic pendulum weighed only about 56 lbs. [1, p. 6] and thus was suitable only for measuring the speed of musket balls or other light shot. Subsequent researchers, such as Charles Hutton in England and General Didion in France, used progressively larger pendulums (one weighing about 6000 kg) to measure the speeds of larger projectiles, up to about 50 lbs [1, pp. 1-11]. Electric chronographs, one of which was invented by Francis Bashforth in 1864, enabled the times required by a projectile to travel successive and equal distances to be precisely determined, hence allowing muzzle speeds and air resistance to be calculated quite exactly [1, pp. 14-26]. In this way, Bashforth and others, most notably General Mayevski in Russia and the Gâvre Commission in France, obtained expressions for air resistance as a function of speed from experimental firings (Bashforth's are described in [1, pp. 27-67]). From the late nineteenth century up to World War I, the following representation for air resistance was widely used for ballistics calculations in the United States and elsewhere [3, pp. 17-28]:

$$f(y,V) = H(y)VG(V)/C, \quad H(y) = e^{-.00003158\,y}, \quad C = m/id^2, \quad G(V) = kV^{n-1}, \qquad (3)$$

where $f(y,V)$ is the magnitude of the force of resistance per unit mass acting on a projectile at altitude *y* feet above sea level and moving at speed *V*, $H(y)$ is a factor that adjusts for lower air density at higher altitude, *C* is the ballistics coefficient determined by the mass *m* of the projectile and its maximum diameter *d* as well as a form factor *i* chosen to make the resistance formula fit projectiles of different shapes, and $G(V)$ is an empirical drag function whose parameters *n* and *k* vary with *V* as in the table below.

V (ft/s)	*n*	$\log_{10} k$
[0, 790]	2	-4.33011
[790, 970]	3	-7.22656
[970, 1230]	5	-13.19813
[1230, 1370]	3	-7.01910
[1370, 1800]	2	-3.88074
[1800, 2600]	1.7	-2.90380
[2600, 3600]	1.55	-2.39095

This drag function $G(V)$ is continuous and piecewise smooth. It was determined in 1883 by Mayevski to fit experimental data obtained by the Krupp firm in Germany (as well as, according to [1, pp. 140-154], unacknowledged data originating with Bashforth). It is clear from (3) and the table above that the square-law of resistance is quite valid for sufficiently low speeds but that, as the speed approaches that of sound in air (1117 ft/s at the standard temperature of 59º F), higher powers of V are needed to represent resistance; remarkably, the square-law (but with a considerably larger coefficient) roughly holds again for a range of supersonic speeds, and lower powers prevail at even higher speeds. Why, one might wonder, did Mayevski and Bashforth choose powers of V to represent the drag function? The answer lies in Johann Bernoulli's solution, considered next.

3. Controversy, Bernoulli's Solution, and Related Work

Consider a projectile in a uniform, downward gravitational field and subject to a resisting force, due to the air through which it moves, assumed to act in a direction opposite to that of the projectile's velocity and to depend only on the projectile's speed (i.e. the projectile's trajectory is presumed low enough that the density of air is roughly constant). By Newton's second law of motion [19, p. 416],

$$\frac{dx}{dt}=u\,,\quad \frac{dy}{dt}=v\,,\quad \frac{du}{dt}=-\frac{f(V)}{V}u\,,\quad \frac{dv}{dt}=-g-\frac{f(V)}{V}v\,, \tag{4}$$

where x and y are the Cartesian coordinates of the projectile, u and v are the components of its velocity in the x and y directions, $V=\sqrt{u^2+v^2}$ is its speed, $f(V)$ is the magnitude of the resisting force on it per unit mass, t is time, and g is the acceleration of gravity. When $f(V)=kV$, i.e. resistance is proportional to the speed, the trajectory satisfying (4) and the initial conditions $x=0$, $y=0$, $u=a$, $v=b$ is

$$y=\left(b+\frac{g}{k}\right)\frac{x}{a}+\frac{g}{k^2}\log\left(1-\frac{kx}{a}\right). \tag{5}$$

Newton constructed this trajectory (see §2) geometrically in the *Principia* (Book 2, Proposition 4); he did not provide an algebraic expression for it like that given in (5), although he remarked that it "is easy to draw … with the help of a table of logarithms" [19, p. 638]. As mentioned above, Newton also partially analyzed projectile motion when resistance varies as the square of the speed (Book 2, Proposition 10), concluding that the trajectory in this case is like a hyperbola with asymptotes at each end, but he was unable to provide an exact construction for the curve.

The story surrounding Johann Bernoulli's solution for the motion of a projectile subject to resistance proportional to the square of the speed is a fascinating one, told in part by I. Bernard Cohen in [19, pp. 168-171] and by A. Rupert Hall in [12, p. 140, pp. 152-156]. Bernoulli discovered a serious error in Newton's analysis for Proposition 10 of Book 2 in the first edition of the *Principia* and communicated this error to Newton via his nephew, Niklaus Bernoulli, when the latter was visiting London in September 1712; the second edition was in press at the time, so Newton worked quickly to rewrite the proposition and to tailor its length so that the revised text occupied the same space as the flawed version that had already been printed [19, pp. 168-169]. Newton did not acknowledge Bernoulli's help in correcting the second edition of the *Principia*; Bernoulli and his supporters charged that Newton did not fully understand his own method of fluxions, which is used extensively in Proposition 10, especially in the form of infinite series; Newton and his supporters made counter-charges [19, pp. 170-171]. The controversy fed the calculus priority dispute that had been raging since 1699 [12, p. 152-154]. In a letter that reached Bernoulli early in 1718, one of Newton's supporters, John Keill, Savilian Professor of Astronomy at Oxford, challenged the former to "Find the curve which a projectile describes through the air on the simplest hypothesis of uniform gravity and density in the medium, the resistance varying as the square of the velocity" [12, p. 155], a challenge based on the problem addressed in Proposition 10 and Newton's failure to completely solve the problem using his method of fluxions in that proposition. Keill evidently thought the challenge to be unanswerable and must have felt surprised and disappointed when Bernoulli produced a complete (granting quadratures) and more general solution.

Bernoulli's solution is given in [2] following his lengthy account of the events associated with it. He ridicules Keill and the challenge, questioning the purpose behind it; he writes that he found his solution soon after receiving the challenge and had then issued a counter-challenge to Keill to solve (granting quadratures) the more general problem he had solved involving resistance varying as an arbitrary power of the speed; he notes that Keill did not respond to the counter-challenge but that English mathematician Brook Taylor did submit a solution after Bernoulli's deadline; he dwells on his triumph over English mathematicians [12, pp. 155-156]. Bernoulli gives his solution only as

$$x = \int z\,Z^{-1/n}\,dz\,, \quad y = \int a\,Z^{-1/n}\,dz\,, \quad Z = \int (a^2 + z^2)^{n-\frac{1}{2}}\,dz\,, \tag{6}$$

where x and y are the coordinates of the projectile's trajectory (the meanings of x and y here are the reverse of those in equation (4) above), $2n$ is the power of the speed to which resistance is proportional (so $n = 1$ in Keill's challenge), and a is an unspecified physical constant (involving the acceleration of gravity, the coefficient of resistance, and n). Bernoulli withholds his derivation of (6), saying that "it is sufficient that I indicate the rule on which it is founded: I published that rule in *Actis Lips.* 1713, page 118 line 5 and page 119 line 2" [2, translated by my colleague David Dahle]. However, Bernoulli does provide the general solution, with derivation, of a relative, a certain Professor Patavin, who solved Keill's original problem and, on learning that Bernoulli had found the general solution, "immediately responded with his own general solution" [2, Dahle]. Suggesting in this way the ease with which problems are solved using Leibniz's calculus compared to that of Newton, Bernoulli is asserting (with good reason) the superiority of the former.

An equivalent but more detailed solution by Bernoulli is given in [20, pp. 95-96], but I could not locate the original source (since "Following John Bernoulli, 1721" is all Routh says on the matter). Letting $f(V) = kV^n$ (i.e. air resistance varies as speed to the power n, not $2n$ as in (6)) and $p = v/u$ in (4) and using the initial conditions $x = 0$, $y = 0$, $u = a$, $v = b$ at $t = 0$ yields the solution:

$$u^{-n} = a^{-n} - kng^{-1}\int_{b/a}^{p} \left(1 + p^2\right)^{(n-1)/2} dp\,, \tag{7}$$

$$t = -g^{-1}\int_{b/a}^{p} u\,dp, \quad x = -g^{-1}\int_{b/a}^{p} u^2 dp, \quad y = -g^{-1}\int_{b/a}^{p} pu^2 dp\,. \tag{8}$$

The integral in (7) can be expressed in elementary terms for any positive integer n, thus yielding an explicit expression for u in terms of p that can be used in the integrals in (8), which can be determined only by numerical quadrature (unless $n = 1$, in which case (5) emerges from (7) and (8)); p can be regarded as a parameter in terms of which t, x, and y are expressed.

In order to give the reader a realistic feel for the effects of air resistance, an example will now be presented. Imagine a standard shot-put projected at 170 m/s (557.74 ft/s, about half the speed of sound in air) and an angle of 45º. The terminal speed V_T of a shot-put (i.e. the maximum speed it acquires while falling through air) is about 145 m/s [17], from which the coefficient k of air resistance when the function $f(V) = kV^n$ is used in (4) is readily found using the fact that $V_T = \sqrt[n]{g/k}$. The projection speed is low enough in this example (see the table in §2) that resistance to the motion is accurately modelled by kV^2 (quadratic drag), but motion resisted as kV (linear drag) will also be considered here as well as motion on Galileo's parabola $y = bx/a - gx^2/2a^2$ (no drag) and an elementary approximation to motion under quadratic drag valid when the inclination angle θ of the projectile's velocity vector (note that $\tan\theta = v/u = p$) is not too large, specifically

$$y = \left(b + \frac{g}{2ka}\right)\frac{x}{a} + \frac{g}{4k^2a^2}\left(1 - e^{2kx}\right). \tag{9}$$

This approximation dates back at least to [15, p. 297], and it is shown in [11] that the curve described by (9) lies between the exact trajectory for quadratic drag and Galileo's parabola for any non-vertical angle of projection. Figure 2 depicts all four trajectories.

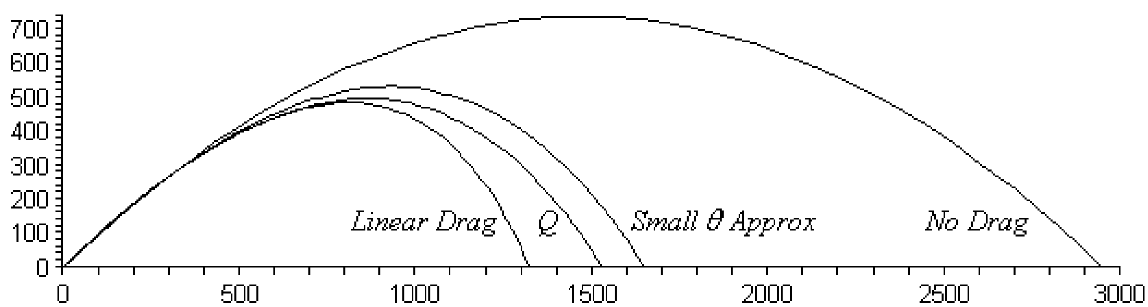

Figure 2: Trajectories of a shot-put projected at 170 m/s and an angle of 45° for linear drag, quadratic drag (Q), the small inclination approximation to quadratic drag, and no drag.

Because the speed of the shot-put is fairly close to its terminal speed on the trajectories for linear and quadratic drag in figure 2 and coefficients of air resistance were calculated from the terminal speed, these trajectories do not differ greatly in figure 2; generally, they differ far more. In any case, it is evident from figure 2 that air resistance makes an enormous difference in the range of a projectile even at moderate initial speeds.

Much of the history of ballistics after the publication of Bernoulli's solution consisted of attempts to calculate approximate trajectories based on it. As noted above, Euler translated Robins' *New Principles* into German, but he also systematically analyzed Bernoulli's solution for the case of quadratic drag [7]. A method based on [7] is described in [18, pp. 305-310]. This method relies on three functions, T, X, and Y, which are identical to t, x, and y, respectively, in (8), up to multiplicative constants. These functions depend only on the inclination angle θ at a point on a trajectory and the angle θ_a of the slant asymptote for the ascending part of the trajectory. When these functions are tabulated with each row corresponding to one value of θ_a and equally-spaced values of θ from θ_a to $-\pi/2$ (the angle of the vertical asymptote for the descending part), the three rows (one for each function) corresponding to the same value of θ_a can be used to find the range, time of flight, maximum altitude, impact speed, and impact angle for the entire species of trajectories having the same slant asymptote. Euler himself used the trapezoidal rule to tabulate the functions for one particular species (corresponding to $\theta_a = 55°$) as an example in [7]. In 1764, H. F. Graevenitz, a German infantry officer, extended Euler's calculations to 18 species [22]. Euler's method was employed as recently as World War II [18, p. 258] and was particularly useful for weapons with relatively low muzzle speeds and high launch angles, such as trench mortars [18, p. 305].

For high muzzle speeds, Euler's method is not suitable (since it assumes quadratic drag), and something like (3) employing an empirical drag function $G(V)$ must be used. When $G(V)$ is described by a table like the one associated with (3), accurate results can always be obtained by dividing the trajectory into small arcs, on each of which the projectile's speed lies in an interval associated with a specific value of n, and integrating Bernoulli's solution numerically on each small arc [5, pp. 49-50]. This small-arc method requires extensive computations, however, and less time-consuming alternatives were sought. One such alternative was the method of Francesco Siacci, a professor at the University of Turin as well as the Turin Military Academy. Siacci's method (ca. 1880), described in [3, pp. 27-41], is accurate for trajectories with low launch angles (up to about 15°) and involves four functions, named S, T, I, and A (for space, time, inclination, and altitude), defined by integrals involving $G(V)$. These functions, which are independent of the trajectory and can therefore be tabulated beforehand, are used to determine the significant quantities (e.g. range, time of flight) of a firing. Siacci's method was widely used up to World War I when higher launch angles became common [3, p. 27]; it was adapted for use in the United States by Col. James Ingalls of the U. S. Army, who computed his own ballistic tables and extended Siacci's method [14]. While employed as a second lieutenant with the Royal Garrison Artillery during World War I, J. E. Littlewood [16] devised a method similar to Siacci's. The error in the range estimates produced by Littlewood's method is $O(\phi^6)$, versus $O(\phi^4)$ for Siacci's method, where ϕ is the launch angle, and Littlewood's method compensated for the effects of reduced air density at greater altitude as well. Littlewood also formulated a method for anti-aircraft fire.

4. Mathematics and its Interaction with Society

G. H. Hardy wrote that ballistics is "repulsively ugly and intolerably dull; even Littlewood could not make ballistics respectable" and that "Real mathematics has no effects on war. … The trivial mathematics … has many applications in war" [13, pp. 140-141]. For Hardy, "the trivial mathematics is, on the whole, useful" [13, p. 139], but the mathematics of ballistics is both trivial and abhorrent because it is also harmful.

Hardy's remarks beg some questions. What is (real) mathematics? Who are (real) mathematicians? How should (real) mathematics and (real) mathematicians interact with society? These questions are both aesthetic and moral, and we cannot expect the same answers from everyone. As we have seen, many real mathematicians, including Galileo, Euler, and Littlewood, found ballistics research aesthetically and morally acceptable. There are other notable examples. In January 1918, Oswald Veblen, a prominent member of the mathematics department at Princeton, began serving as the commander of the office of experimental ballistics at the U. S. Army's new Aberdeen Proving Ground [9]. His main role there was to supervise the production of range tables for new guns acquired by the army; in addition to experimental firings, this required a thorough knowledge of current ballistics theory (largely Siacci's) and laborious calculation, carried out by a growing staff of human computers, one of whom was Norbert Wiener [9]. Wiener later wrote that "For many years after the First World War, the overwhelming majority of significant American mathematicians was to be found among those who had gone through the discipline of the Proving Ground" [10, p. 152]. Veblen also worked with University of Chicago astronomer Forest Ray Moulton, who directed the office of mathematical ballistics in Washington, to develop a new theory of ballistics [9].

Ballistics is the fascinating product of the interaction between technology, government, the military, science, mathematics, and society in general. It was born of new technologies, its development encouraged by kings, its creations exploited by generals, its principles studied by scientists for whom the nature of motion was primary, its equations solved by mathematicians contending over calculus, and its results visited upon ordinary people. Much of this interaction during the eighteenth and nineteenth centuries is well documented in [22]: the research of Robins, Euler, and others was soon incorporated into the curricula of military schools and universities; the military employed carefully packaged ballistics theory on the battlefield; governments established proving grounds for testing new weapons and determining drag functions. Large-scale organizational computing, one of the social effects of ballistics research, is treated nicely in [10, pp. 126-158]; because of the difficulty in hiring men during the war, many of the human computers hired by Moulton at his ballistics office in Washington were women who had been mathematics majors at university. The role of women in organizational scientific computing grew enormously thereafter, until computing of this kind was replaced by electronic computers such as the ENIAC, which was itself invented primarily to compute ballistic tables [10, pp. 276-317]. Ballistics research, as well as the government and military funding available to mathematicians who engage in it, continue to affect the mathematics community and cause division within it. I well recall, for example, the debate within the community in the 1980s over applying for funds associated with President Ronald Reagan's Strategic Defence Initiative. Mathematicians who, like G. H. Hardy, question ballistics research on moral grounds seem to be far more numerous now than they were during the First World War. That is definitely a good thing.

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